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Direct and Inverse Cascades in Two-Dimensional Turbulence with a Generalized Enstrophy Invariant

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Direct and reverse cascades are shown to be generic to two-dimensional turbulence with a generalized enstrophy invariant $\tilde{U} = \int \frac{1}{2}|\nabla^{2q}\Psi|^2\, dx = \text{const}$, $q \neq \frac{1}{2}$. The classical hydrodynamical situation is a special case ($q = 1$) of the general result.

1. INTRODUCTION

Two-dimensional hydrodynamic turbulence is characterized by the two following underlying integral invariants:

$$\text{energy } W \equiv \int \frac{1}{2} |\nabla\Psi|^2\, dx = \text{const} \tag{1}$$

$$\text{enstrophy } U \equiv \int \frac{1}{2} |\nabla^2\Psi|^2\, dx = \text{const} \tag{2}$$

where $\Psi(x, y)$ is the stream function. It was conjectured by Kraichnan [1] and Batchelor [2] that these two invariants give rise to two cascades, an inverse energy cascade, in which energy propagates toward larger scales, and a direct enstrophy cascade, in which enstrophy propagates toward smaller scales.

Numerical calculations of Frisch *et al.* [3], Lilly [4], Herring *et al.* [5], Pouquet *et al.* [6], Fornberg [7], and Frisch and Sulem [8], among others, and laboratory experiments of Couder [9], Sommeria [10], Rutgers [11], and Paret and Tabeling [12], among others, confirmed the conjecture of Kraichnan [1] and Batchelor [2]. When energy and enstrophy are continuously injected at a fixed wavenumber, a quasi-steady regime was found to develop where

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enstrophy cascades to large wavenumbers across a k^{-3} inertial range with zero energy transfer, while energy flows indefinitely to small wavenumbers across a $k^{-5/3}$ inertial range with zero enstrophy transfer. The purpose of this paper is to complement these investigations by giving an analytical demonstration of the inverse energy cascade and direct enstrophy cascade and to show that these results are indeed generic to a two-dimensional turbulence with a generalized enstrophy invariant

$$\tilde{U} \equiv \int \frac{1}{2} |\nabla^{2q}\Psi|^2 \, d\mathbf{x} = \text{const}, \qquad q \neq \frac{1}{2} \tag{3}$$

[The case $q = 1/2$ is to be excluded because, for this case, (3) simply degenerates into (1)!] The classical hydrodynamic case corresponds to $q = 1$.

2. ENERGY AND GENERALIZED ENSTROPHY CASCADES

Consider a source in the spectral space at wavenumber k_s with energy $W_s = W(k_s)$. This source would then decay via triadic interactions into two modes with wavenumbers k_1 and k_2 and energies W_1 and W_2 respectively. Since energy and generalized enstrophy are conserved during this decay, we have

$$W_s = W_1 + W_2 \tag{4}$$

$$k_s^{4q-2} W_s = k_1^{4q-2} W_1 + k_s^{4q-2} W_2 \tag{5}$$

(4) and (5) imply that the energy is partitioned as follows:

$$W_1 = \frac{k_2^{4q-2} - k_s^{4q-2}}{k_2^{4q-2} - k_1^{4q-2}} W_s \tag{6a}$$

$$W_2 = \frac{k_s^{4q-2} - k_1^{4q-2}}{k_2^{4q-2} - k_1^{4q-2}} W_s \tag{6b}$$

(6a) and (6b) imply in turn that

$$k_2^{4q-2} > k_s^{4q-2} > k_1^{4q-2} \tag{7}$$

so that k_s lies between k_1 and k_2.

Now, (4) implies that

$$W_1 = pW_s, \qquad W_2 = (1 - p)W_s, \qquad 0 < p < 1 \tag{8}$$

If we put further

$$k_1 = \alpha^{1/(4q-2)} k_s, \qquad k_2 = \beta^{1/(4q-2)} k_s \tag{9}$$

we have from (5),

$$\alpha W_1 + \beta W_2 = W_s = W_1 + W_2 \tag{10}$$

(8) and (10) then lead to

$$(\alpha - 1)p + (\beta - 1)(1 - p) = 0 \tag{11}$$

from which

$$\alpha = p, \qquad \beta = 1 + p \tag{12}$$

(12) was proposed (without deduction) by Hasegawa [13] in the special case $q = 1$.

Let us therefore assume that a mode k_s first decays into modes k_1 ($k_1 = p^{1/(4q-2)}k_s$) and k_2 ($k_2 = (1 + p)^{1/(4q-2)}k_s$) with corresponding energies $W_1 = pW_s$ and $W_2 = (1 - p)W_s$ and generalized enstrophies $\tilde{U}_1 = k_s^{4q-2}p^2W_s$ and $\tilde{U}_2 = k_s^{4q-2}(1 - p^2)W_s$, respectively.

In the next step of the cascade, the mode k_1 decays into a mode $p^{1/(4q-2)}k_1 = p^{1/(2q-1)}k_s$ and another mode $(1 + p)^{1/(4q-2)}k_1 = p^{1/(4q-2)}(1 + p)^{1/(4q-2)}k_s$, while the mode k_2 decays into a mode $p^{1/(4q-2)}k_2 = p^{1/(4q-2)}(1 + p)^{1/(4q-2)}k_s$ and another mode $(1 + p)^{1/(4q-2)}k_2 = (1 + p)^{1/(2q-1)}k_s$. The energies of the modes $p^{1/(2q-1)}k_s$, $p^{1/(4q-2)}(1 + p)^{1/(4q-2)}k_s$, and $(1 + p)^{1/(2q-1)}k_s$ are p^2W_s, $2p(1 - p)W_s$, and $(1 - p)^2W_s$, respectively.

Thus, at the nth step of the cascade, the energy is given by

$$W(k = p^{(n-r)/(4q-2)}(1 + p)^{r/(4q-2)}k_s) = \binom{n}{r}p^{n-r}(1 - p)^rW_s \tag{13}$$

Now, by the de Moivre–Laplace approximation, we have for the binomial distribution [14]

$$\lim_{n\Rightarrow\infty}\binom{n}{r}p^{n-r}(1 - p)^r \approx \frac{1}{\sqrt{2\pi np(1 - p)}}\, e^{(n-r-np)^2/2\pi p(1-p)} \tag{14}$$

so that the binomial distribution in (13) peaks at $r/n \approx 1 - p$ as $n \Rightarrow \infty$. The corresponding wavenumber is given by

$$\begin{aligned} k_* &= \lim_{n\Rightarrow\infty}[p^{n-r}(1 + p)^r]^{1/(4q-2)}k_s \\ &= \lim_{n\Rightarrow\infty}[p^{1-(r/n)}(1 + p)^{r/n}]^{n/(4q-2)}k_s \\ &= \lim_{n\Rightarrow\infty}[p^p(1 + p)^{1-p}]^{n/(4q-2)}k_s \end{aligned} \tag{15}$$

In order to evaluate the limit in (15), it proves to be convenient to use the following result [15] on the generalized arithmetic/geometric/harmonic means (I am deeply indebted to Dr. M. D. Taylor for this perceptive suggestion):

Suppose $a_1, \ldots, a_n$ and $p_1, \ldots, p_n$ are positive numbers such that $p_1 + \cdots + p_n = 1$ and $a_i \neq a_j$ $(i \neq j)$; then,

$$\frac{1}{p_1/a_1 + \cdots + p_n/a_n} < a_1^{p_1} \cdots a_n^{p_n} < p_1 a_1 + \cdots + p_n a_n \tag{16}$$

Taking $n = 2$, $a_1 = p$, $a_2 = 1 + p$, $p_1 = p$, and $p_2 = 1 - p$, we obtain from the second inequality

$$p^p(1 + p)^{1-p} < p^2 + (1 - p^2) = 1, \qquad 0 < p < 1 \tag{17}$$

Using (17), we obtain from (15)

$$k_* \approx 0 \tag{18}$$

Therefore, as $n \Rightarrow \infty$ the peak of the energy distribution moves to $k \Rightarrow 0$, and the energy cascades inversely even in the general case.

Next, the generalized enstrophies of the modes $p^{1/(2q-1)}k_s$, $p^{1/(4q-2)}(1 + p)^{1/(4q-2)}k_s$, and $(1 + p)^{1/(2q-1)}k_s$ are $k_s^{4q-2}p^4 W_s$, $2k_s^{4q-2}p^2(1 - p^2)W_s$, and $k_s^{4q-2}(1 - p^2)^2)W_s$, respectively.

Thus, at the nth step of the cascade, the generalized enstrophy is given by

$$\tilde{U}(k = p^{(n-r)/(4q-2)}(1 + p)^{r/(4q-2)}k_s) = \binom{n}{r}(p^2)^{n-r}(1 - p^2)^r k_s^{4q-2} W_s \tag{19}$$

Once again, by using the de Moivre–Laplace approximation, we see that the binomial distribution in (19) peaks at $r/n \approx 1 - p^2$ as $n \Rightarrow \infty$. The corresponding wavenumber is given by

$$\begin{aligned} \tilde{k}_* &= \lim_{n\Rightarrow\infty}[p^{n-r}(1 + p)^r]^{1/(4q-2)}k_s \\ &= \lim_{n\Rightarrow\infty}[p^{1-(r/n)}(1 + p)^{r/n}]^{n/(4q-2)}k_s \\ &= \lim_{n\Rightarrow\infty}[p^{p^2}(1 + p)^{1-p^2}]^{n/(4q-2)}k_s \end{aligned} \tag{20}$$

In order to evaluate the limit in (20), take $n = 2$, $a_1 = p$, $a_2 = 1 + p$, $p_1 = p^2$, and $p_2 = 1 - p^2$; we then obtain from the first inequality in (16),

$$p^{p^2}(1 + p)^{p^2} > \frac{1}{p^2/p + (1 - p^2)/(1 + p)} = 1, \qquad 0 < p < 1 \tag{21}$$

Using (21), we obtain from (20)

$$\tilde{k}_* \Rightarrow \infty \tag{22}$$

Therefore, the peak of the generalized enstrophy distribution moves to $k \Rightarrow \infty$ as $n \Rightarrow \infty$, and the generalized enstrophy cascades directly.

3. DISCUSSION

In this paper, we have given an analytical demonstration of the inverse energy cascade and direct enstrophy cascade in two-dimensional turbulence and have shown that these results are stronger than what may appear from a consideration of the classical hydrodynamic case. They are indeed generic to a two-dimensional situation with a generalized enstrophy invariant

$$\tilde{U} \equiv \int \frac{1}{2} |\nabla^{2q}\Psi|^2 \, dx = \text{const}, \qquad q \neq \frac{1}{2} \tag{3}$$

of which the classical hydrodynamic situation is a special case ($q = 1$)!

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